Waves, the Wave Equation, and Phase Velocity

What is a wave?

Forward [$f(x-vt)$] and backward [$f(x+vt)$] propagating waves

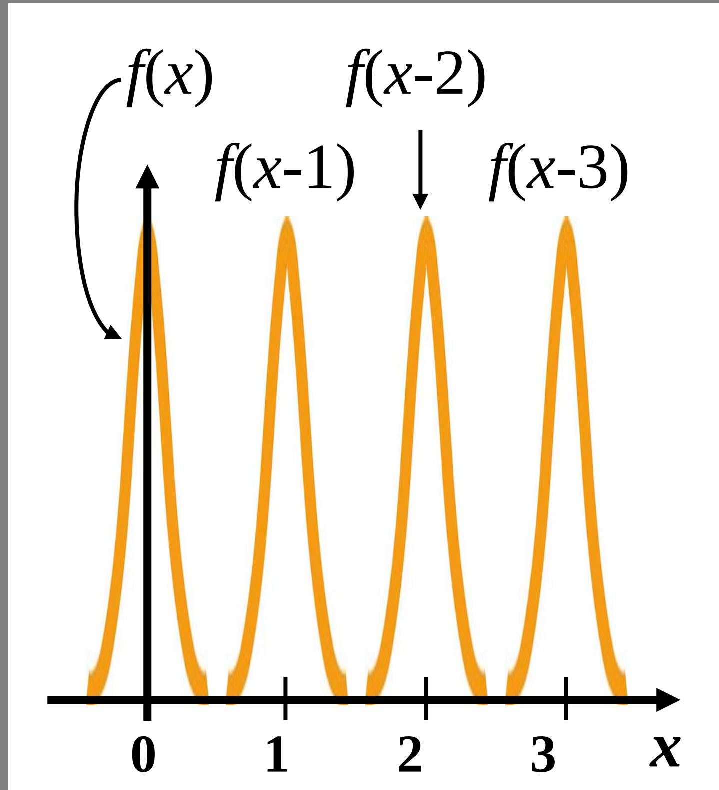
The one-dimensional wave equation

Wavelength, frequency, period, etc.

Phase velocity   Complex numbers

Plane waves and laser beams   Boundary conditions

Div, grad, curl, etc., and the 3D Wave equation



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What is a wave?

A wave is anything that moves.

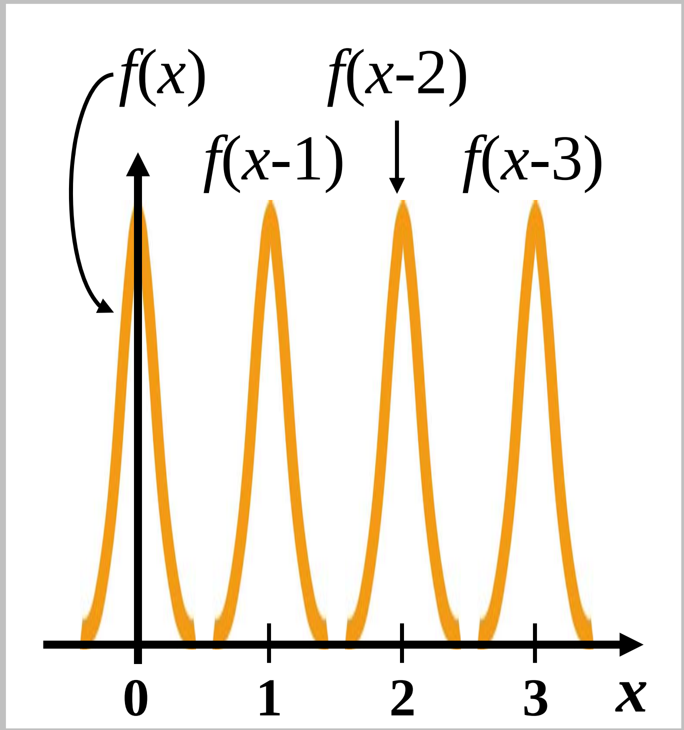
To displace any function $f(x)$ to the right, just change its argument from x to $x\text{-}a$, where a is a positive number.

If we let $a = v\,t$, where v is positive and t is time, then the displacement will increase with time.

So $f(x - v\,t)$ represents a rightward, or forward, propagating wave.

Similarly, $f(x + v\,t)$ represents a leftward, or backward, propagating wave.

v will be the velocity of the wave.



The one-dimensional wave equation

The one-dimensional wave equation for scalar (i.e., non-vector) functions, f:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2} = 0$$

where v will be the velocity of the wave.

The wave equation has the simple solution:

$$f(x,t) = f(x \pm \mathrm{v}t)$$

where $f(u)$ can be any twice-differentiable function.

Proof that $f\,(x \pm \mathrm{v}t)$ solves the wave equation

Write $f(x \pm \mathrm{v}t)$ as $f(u)$, where $u = x \pm \mathrm{v}t$. So $\dfrac{\partial u}{\partial x} = 1$ and $\dfrac{\partial u}{\partial t} = \pm\,\mathrm{v}$

Now, use the chain rule: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}$ $\qquad \dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial t}$

So $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u} \;\Rightarrow\; \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial u^2}$ and $\dfrac{\partial f}{\partial t} = \pm\,\mathrm{v}\dfrac{\partial f}{\partial u} \;\Rightarrow\; \dfrac{\partial^2 f}{\partial t^2} = \mathrm{v}^2\dfrac{\partial^2 f}{\partial u^2}$

Substituting into the wave equation:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2} \;=\; \frac{\partial^2 f}{\partial u^2} - \frac{1}{\mathrm{v}^2}\left\{\mathrm{v}^2\frac{\partial^2 f}{\partial u^2}\right\} \;=\; 0$$

The 1D wave equation for light waves

$$\frac{\partial^2 E}{\partial x^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

where E is the light electric field

We'll use cosine- and sine-wave solutions:

$$E(x,t) = B\cos[k(x \pm \mathrm{v}t)] + C\sin[k(x \pm \mathrm{v}t)]$$

$$\downarrow$$

or

$$kx \pm (k\mathrm{v})t$$

$$\downarrow$$

$$E(x,t) = B\cos(kx \pm \omega t) + C\sin(kx \pm \omega t)$$

where:

$$\frac{\omega}{k} = \mathrm{v} = \frac{1}{\sqrt{\mu\varepsilon}}$$

The speed of light in vacuum, usually called "c", is 3×10^{10} cm/s.

A simpler equation for a harmonic wave:

$$E(x,t) = A\cos[(kx - \omega t) - \theta]$$

Use the trigonometric identity:

$$\cos(z-y) = \cos(z)\cos(y) + \sin(z)\sin(y)$$

where $z = kx - \omega t$ and $y = \theta$ to obtain:

$$E(x,t) = A\cos(kx - \omega t)\cos(\theta) + A\sin(kx - \omega t)\sin(\theta)$$

which is the same result as before,

$$E(x,t) = B\cos(kx - \omega t) + C\sin(kx - \omega t)$$

For simplicity, we'll just use the forward-propagating wave.

as long as:

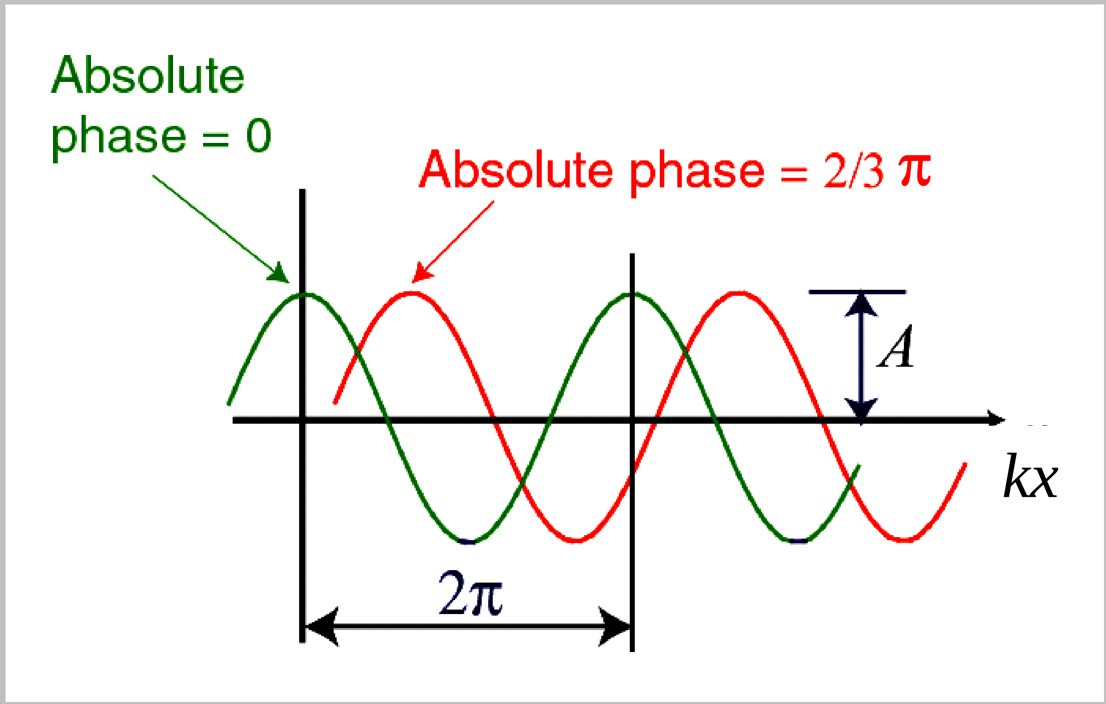
$$A\cos(\theta) = B \quad \text{and} \quad A\sin(\theta) = C$$

Definitions: Amplitude and Absolute phase

$$E(x,t) = A \cos[(k\,x - \omega\,t\,) - \theta\,]$$

A = Amplitude

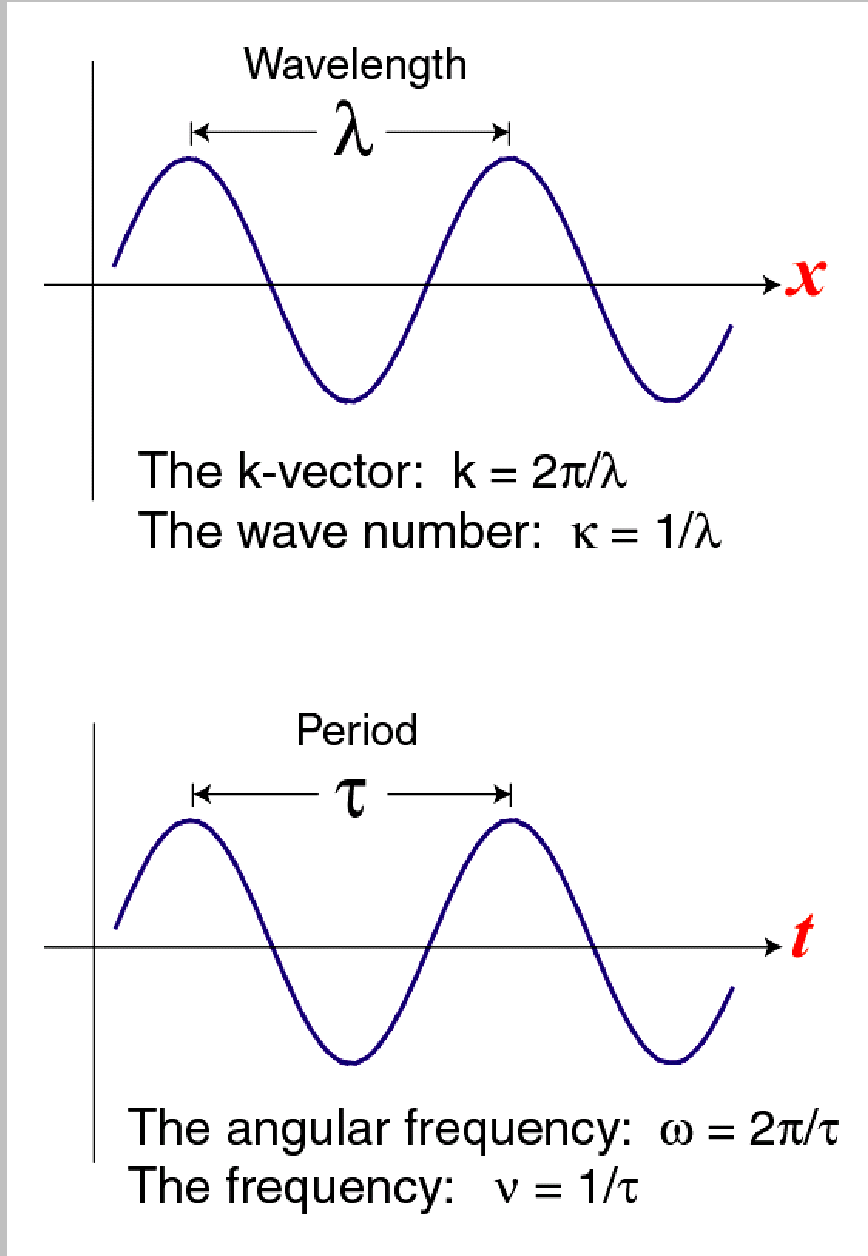
θ = Absolute phase (or initial phase)



Definitions

Spatial quantities:

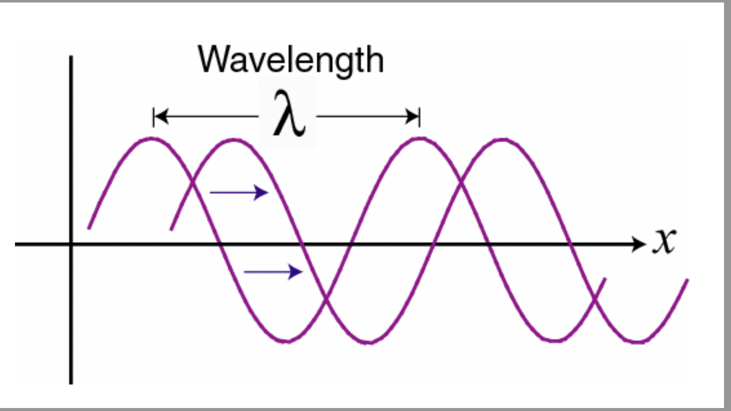
Temporal quantities:



The Phase Velocity

How fast is the wave traveling?

Velocity is a reference distance divided by a reference time.



The phase velocity is the wavelength / period: $\text{v} = \lambda / \tau$

Since $\nu = 1/\tau$:

$$\text{v} = \lambda \nu$$

In terms of the k-vector, $k = 2\pi / \lambda$, and the angular frequency, $\omega = 2\pi / \tau$, this is:

$$\text{v} = \omega / k$$

Human wave

A typical human wave has a phase velocity of about 20 seats per second.

The Phase of a Wave

The phase is everything inside the cosine.

$$E(x,t) = A\cos(\varphi), \text{ where } \varphi = k\,x - \omega\,t - \theta$$

$\varphi = \varphi(x,y,z,t)$ and is not a constant, like θ!

In terms of the phase,

$$\omega = -\partial\varphi/\partial t$$

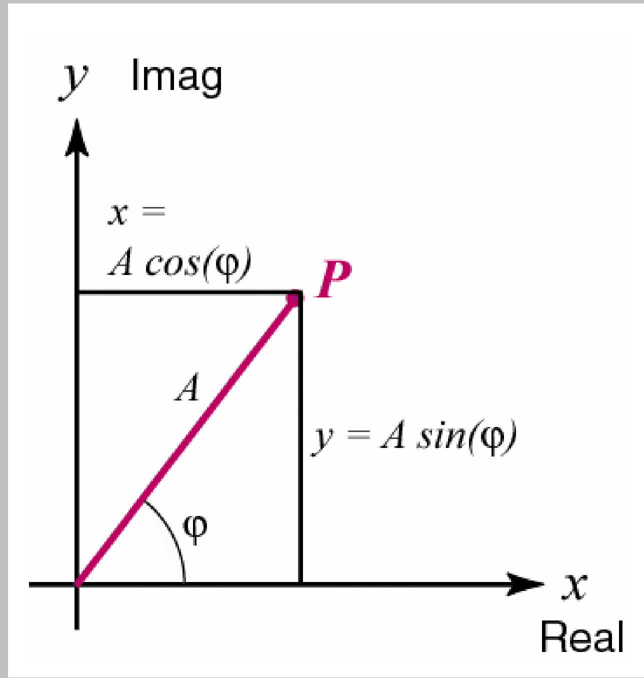
$$k = \partial\varphi/\partial x$$

And

$$\text{v} = \frac{-\partial\varphi/\partial t}{\partial\varphi/\partial x}$$

This formula is useful when the wave is really complicated.

Complex numbers



Consider a point, $P = (x,y)$, on a 2D Cartesian grid.

Let the x-coordinate be the real part and the y-coordinate the imaginary part of a complex number.

So, instead of using an ordered pair, (x,y), we write:

$$P = x + i\,y$$
$$= A\cos(\varphi) + i\,A\sin(\varphi)$$

where $i = (-1)^{1/2}$

Euler's Formula

$$\exp(i\varphi) \;=\; \cos(\varphi) + i\,\sin(\varphi)$$

so the point, $P = A\cos(\varphi) + i\,A\sin(\varphi)$, can be written:

$$P \;=\; A\exp(i\varphi)$$

where

$$A \;=\; \text{Amplitude}$$

$$\varphi \;=\; \text{Phase}$$

Proof of Euler's Formula $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$

Use Taylor Series: $$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$$

If we substitute $x = i\varphi$ into $\exp(x)$, then:

$$\exp(i\varphi) = 1 + \frac{i\varphi}{1!} - \frac{\varphi^2}{2!} - \frac{i\varphi^3}{3!} + \frac{\varphi^4}{4!} + \ldots$$

$$= \left[1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \ldots\right] + i\left[\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \ldots\right]$$

$$= \cos(\varphi) + i\sin(\varphi)$$

Complex number theorems

If $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$

$$\exp(i\pi) = -1$$

$$\exp(i\pi/2) = i$$

$$\exp(-i\varphi) = \cos(\varphi) - i\sin(\varphi)$$

$$\cos(\varphi) = \frac{1}{2}\left[\exp(i\varphi) + \exp(-i\varphi)\right]$$

$$\sin(\varphi) = \frac{1}{2i}\left[\exp(i\varphi) - \exp(-i\varphi)\right]$$

$$A_1\exp(i\varphi_1) \times A_2\exp(i\varphi_2) = A_1A_2\exp\left[i(\varphi_1 + \varphi_2)\right]$$

$$A_1\exp(i\varphi_1) / A_2\exp(i\varphi_2) = A_1/A_2\exp\left[i(\varphi_1 - \varphi_2)\right]$$

More complex number theorems

Any complex number, z, can be written:

$$z = \mathrm{Re}\{ z \} + i\,\mathrm{Im}\{ z \}$$

So

$$\mathrm{Re}\{ z \} = 1/2 \; (z + z^*)$$

and

$$\mathrm{Im}\{ z \} = 1/2i \; (z - z^*)$$

where z^* is the complex conjugate of z ($i \to -i$)

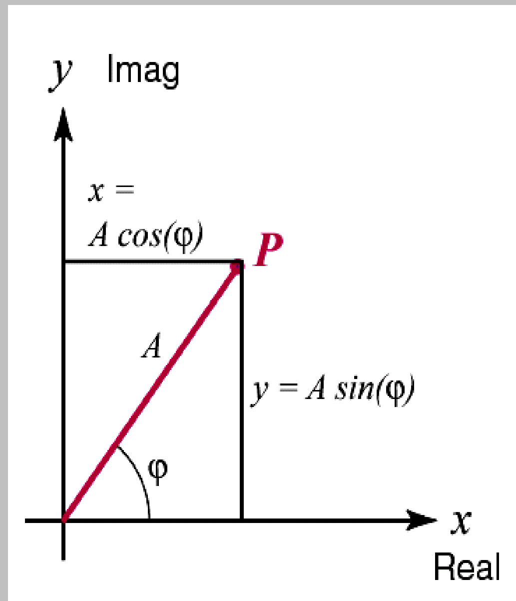
The "magnitude," $| z |$, of a complex number is:

$$| z |^2 = z\, z^* = \mathrm{Re}\{ z \}^2 + \mathrm{Im}\{ z \}^2$$

To convert z into polar form, $A \exp(i\varphi)$:

$$A^2 = \mathrm{Re}\{ z \}^2 + \mathrm{Im}\{ z \}^2$$

$$\tan(\varphi) = \mathrm{Im}\{ z \} / \mathrm{Re}\{ z \}$$



We can also differentiate exp(*ikx*) as if the argument were real.

$$\frac{d}{dx}\exp(ikx) = ik\exp(ikx)$$

Proof :

$$\frac{d}{dx}\left[\cos(kx) + i\sin(kx)\right] = -k\sin(kx) + ik\cos(kx)$$

$$= ik\left[-\frac{1}{i}\sin(kx) + \cos(kx)\right]$$

But $-1/i = i$, so: $= ik\left[i\sin(kx) + \cos(kx)\right]$

Waves using complex numbers

The electric field of a light wave can be written:

$$E(x,t) = A\cos(kx - \omega t - \theta)$$

Since $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$, $E(x,t)$ can also be written:

$$E(x,t) = \mathrm{Re}\,\{\, A\exp[i(kx - \omega t - \theta)]\,\}$$

or

$$E(x,t) = 1/2\, A\exp[i(kx - \omega t - \theta)] + c.c.$$

We often write these expressions without the ½, Re, or +c.c.

where "+ $c.c.$" means "plus the complex conjugate of everything before the plus sign."

Waves using complex amplitudes

We can let the amplitude be complex:

$$E(x,t) = A\exp\left[i\left(kx - \omega t - \theta\right)\right]$$

$$E(x,t) = \left\{A\exp(-i\theta)\right\}\left\{\exp\left[i\left(kx - \omega t\right)\right]\right\}$$

where we've separated the constant stuff from the rapidly changing stuff.

The resulting "complex amplitude" is:

$$\underset{\sim}{E}_0 = A\exp(-i\theta) \qquad \leftarrow \text{(note the " \sim ")}$$

So:

$$\underset{\sim}{E}(x,t) = \underset{\sim}{E}_0 \exp i\left(kx - \omega t\right)$$

As written, this entire field is complex!

How do you know if E_0 is real or complex?

Sometimes people use the "~", but not always.
So always assume it's complex.

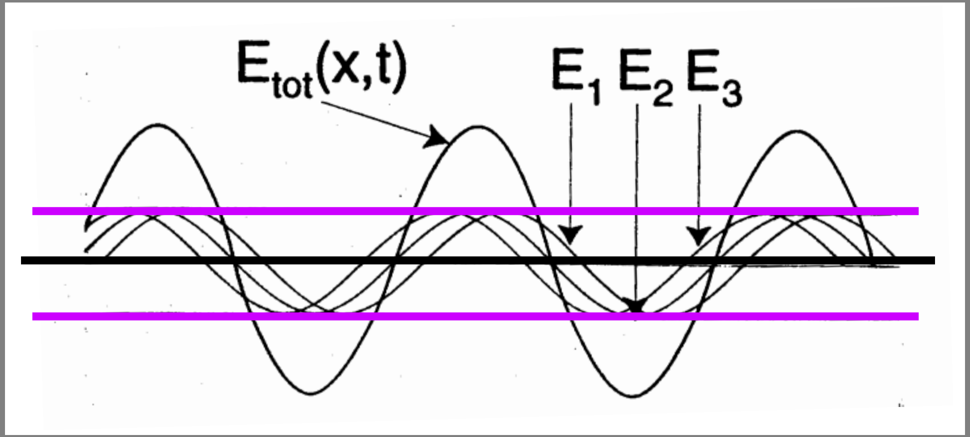
Complex numbers simplify waves!

Adding waves of the same frequency, but different initial phase, yields a wave of the same frequency.

This isn't so obvious using trigonometric functions, but it's easy with complex exponentials:

$$\underset{\sim}{E}_{tot}(x,t) = \underset{\sim}{E}_1 \exp i(kx-\omega t) + \underset{\sim}{E}_2 \exp i(kx-\omega t) + \underset{\sim}{E}_3 \exp i(kx-\omega t)$$
$$= (\underset{\sim}{E}_1 + \underset{\sim}{E}_2 + \underset{\sim}{E}_3) \exp i(kx-\omega t)$$

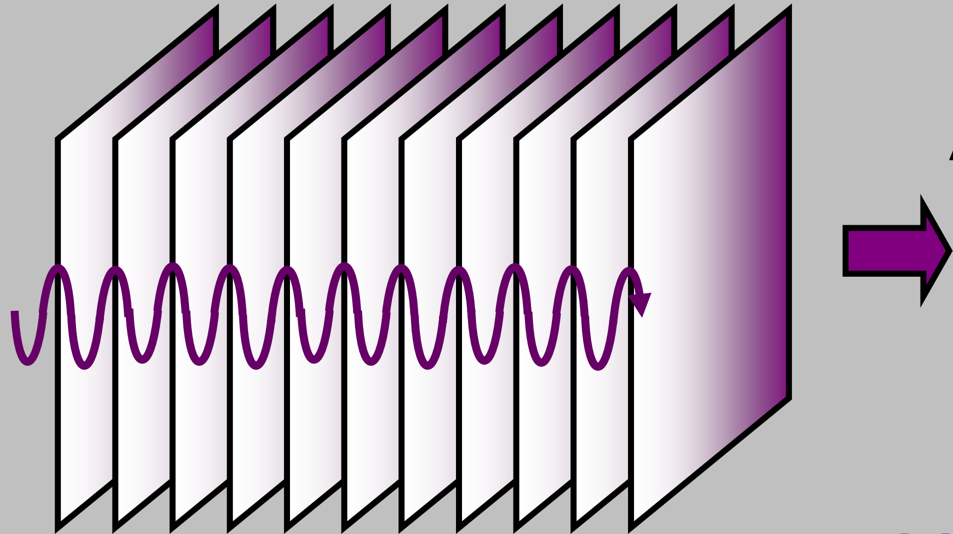
where all initial phases are lumped into E_1, E_2, and E_3.



$\underset{\sim}{E}_0 \exp[i(kx - \omega t)]$ **is called a plane wave.**

A plane wave’s contours of maximum field, called **wave-fronts** or **phase-fronts**, are planes. They extend over all space.

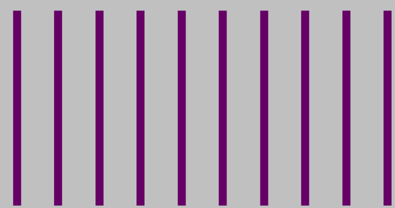
Wave-fronts are helpful for drawing pictures of interfering waves.



A wave's wave-fronts sweep along at the speed of light.

A plane wave's wave-fronts are equally spaced, a wavelength apart.

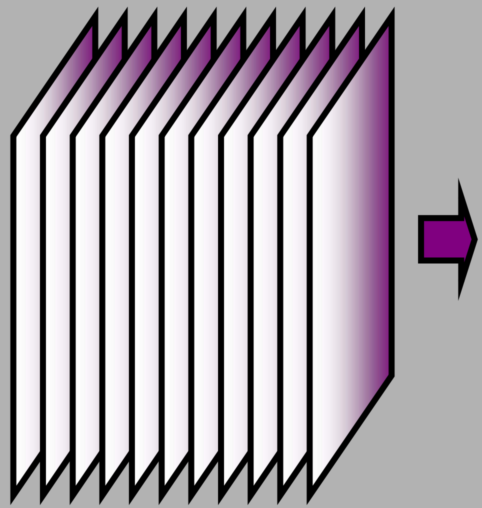
They're perpendicular to the propagation direction.



Usually, we just draw lines; it’s easier.

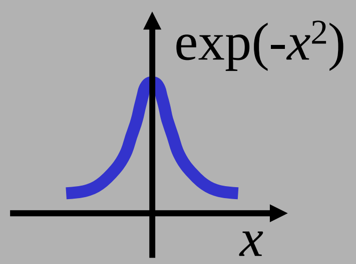
Localized waves in space: beams

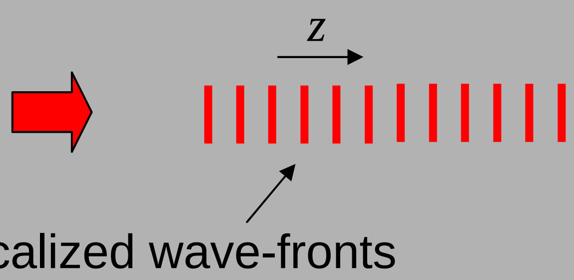
A plane wave has flat wave-fronts throughout all space. It also has infinite energy. It doesn't exist in reality.

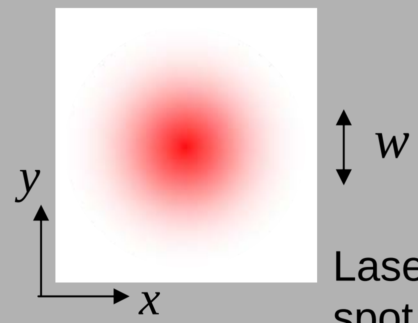


Real waves are more localized. We can approximate a realistic wave as a plane wave vs. z times a Gaussian in x and y:

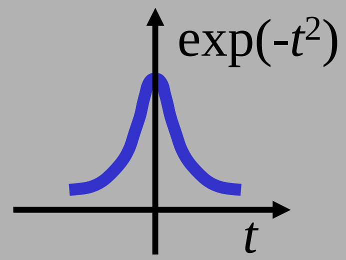
$$\underset{\sim}{E}(x,y,z,t) = \underset{\sim}{E}_0 \exp\left[-\frac{x^2+y^2}{w^2}\right]\exp[i(kz-\omega t)]$$



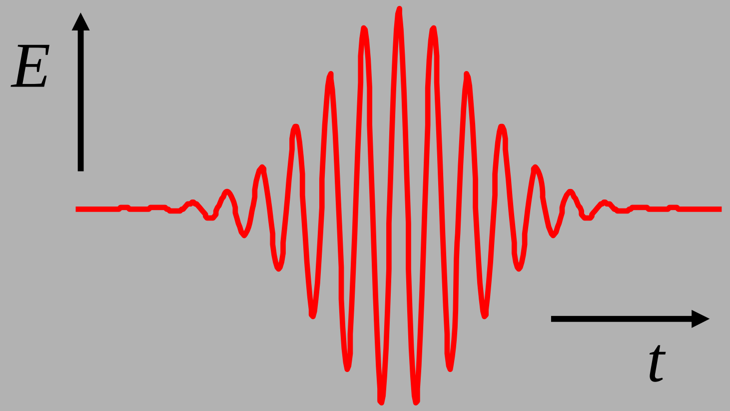




Localized waves in time: pulses



If we can localize the beam in space by multiplying by a Gaussian in x and y, we can also localize it in time by multiplying by a Gaussian in **time**.

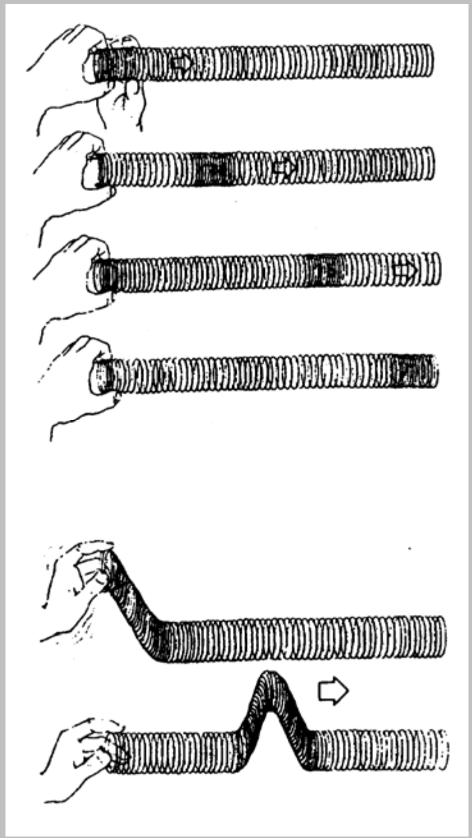


$$\underset{\sim}{E}(x,y,z,t) = \underset{\sim}{E}_0 \exp\left[-\frac{t^2}{\tau^2}\right]\exp\left[-\frac{x^2+y^2}{w^2}\right]\exp[i(kz-\omega t)]$$

This is the equation for a laser pulse.

Longitudinal vs. Transverse waves

Longitudinal:



Motion is along the direction of propagation—**longitudinal polarization**

Transverse:

Motion is transverse to the direction of propagation—**transverse polarization**

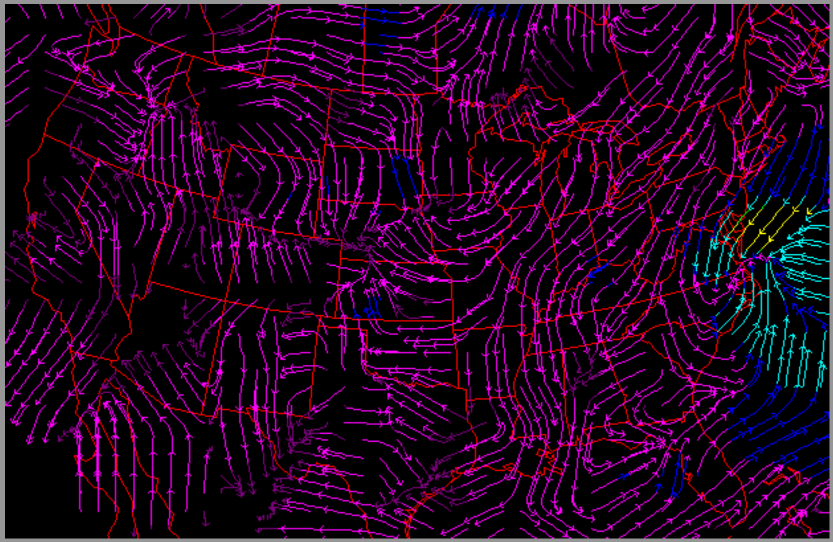
Space has 3 dimensions, of which 2 are transverse to the propagation direction, so there are 2 transverse waves in addition to the potential longitudinal one.

The direction of the wave's variations is called its **polarization**.

Vector fields

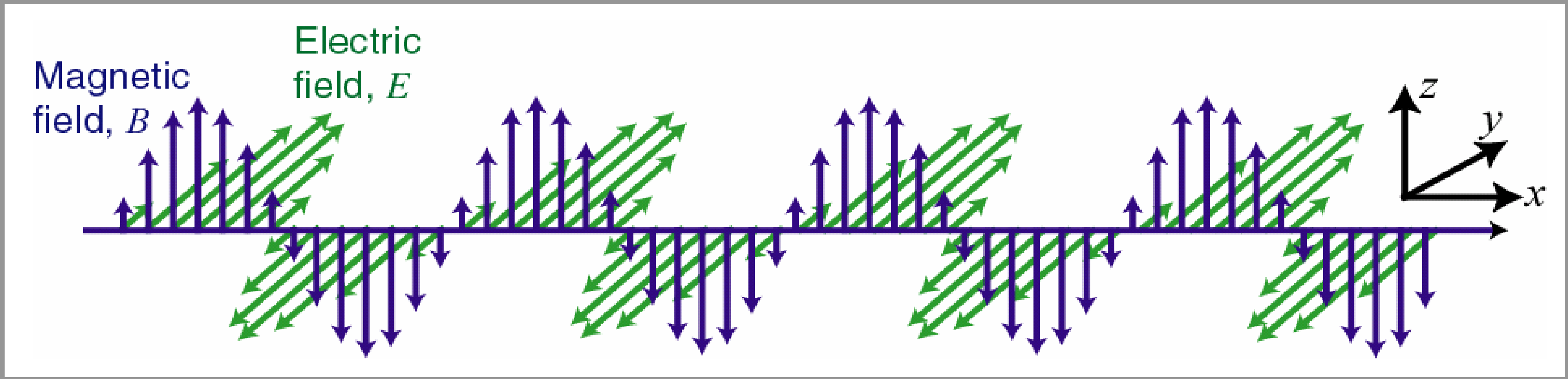
Light is a 3D vector field.

A 3D vector field $\vec{f}(\vec{r})$ assigns a 3D vector (i.e., an arrow having both direction and length) to each point in 3D space.



Wind patterns: 2D vector field

A light wave has both electric and magnetic 3D vector fields:



And it can propagate in any direction.

Div, Grad, Curl, and all that

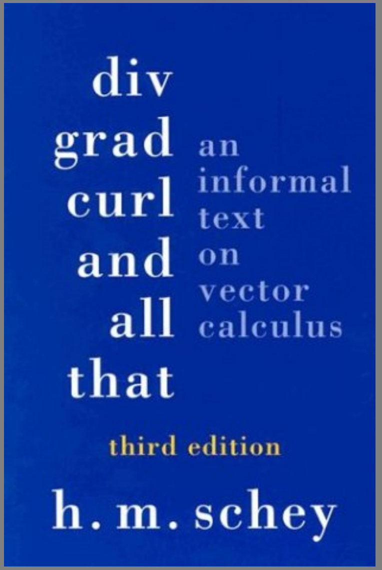
Types of 3D vector derivatives:

The Del operator: $$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} , \frac{\partial}{\partial z} \right)$$

The Gradient of a scalar function f:

$$\vec{\nabla} f \equiv \left(\frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} , \frac{\partial f}{\partial z} \right)$$

The gradient points in the direction of steepest ascent.



If you want to know more about vector calculus, read this book!

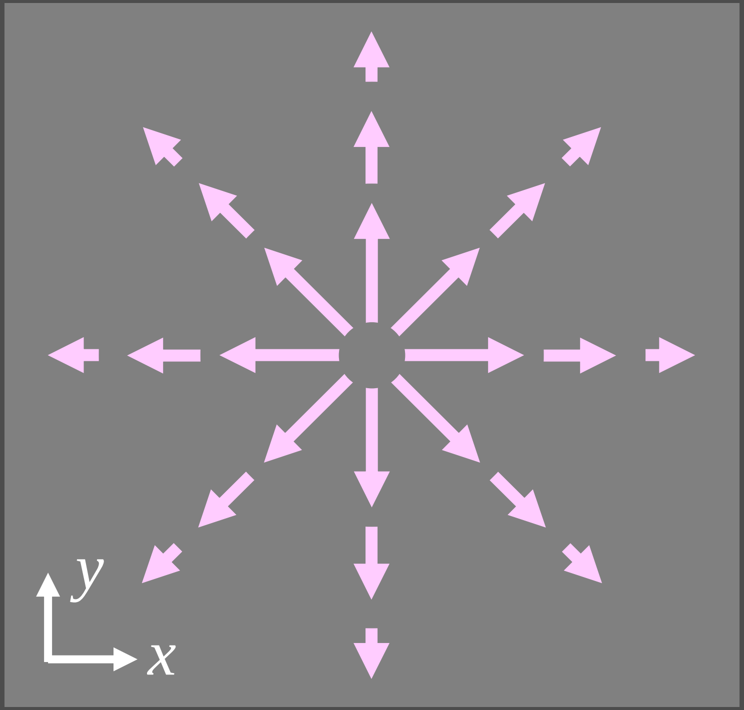
Div, Grad, Curl, and all that

The **Divergence** of a vector function:

$$\vec{\nabla} \cdot \vec{f} \equiv \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

The **Divergence** is nonzero
if there are sources or sinks.

A 2D source with a
large divergence:



Note that the x-component of this function changes rapidly in the x direction, etc., the essence of a large divergence.

Div, Grad, Curl, and more all that

The **Laplacian** of a scalar function :

$$\nabla^2 f \equiv \vec{\nabla}\cdot\vec{\nabla} f = \vec{\nabla}\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The **Laplacian of a vector** function is the same,
but for each component of *f*:

$$\nabla^2 \vec{f} = \left(\frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2}, \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2}, \frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}\right)$$

The Laplacian tells us the curvature of a vector function.

The 3D wave equation for the electric field and its solution

A light wave can propagate in any direction in space. So we must allow the space derivative to be 3D:

$$\vec{\nabla}^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

or

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

whose solution is:

$$E(x,y,z,t) = \mathrm{Re}\left\{ \underset{\sim}{E}_0 \exp[i(\vec{k}\cdot\vec{r} - \omega t)] \right\}$$

where $\vec{k} \equiv \left(k_x, k_y, k_z\right)$ $\quad \vec{r} \equiv \left(x, y, z\right)$

and

$$\vec{k}\cdot\vec{r} \equiv k_x\, x + k_y\, y + k_z\, z$$

$$k^2 \equiv k_x^2 + k_y^2 + k_z^2$$

The 3D wave equation for a light-wave electric field is actually a vector equation.

And a light-wave electric field can point in any direction in space:

$$\vec{\nabla}^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Note the arrow over the E.

whose solution is:

$$\vec{E}(x,y,z,t) = \mathrm{Re}\left\{ \underset{\sim}{\vec{E}_0} \exp[i(\vec{k}\cdot\vec{r} - \omega t)] \right\}$$

where:

$$\underset{\sim}{\vec{E}_0} = (\underset{\sim}{E}_{0x}, \underset{\sim}{E}_{0y}, \underset{\sim}{E}_{0z})$$

Vector Waves

We must now allow the field E and its complex field amplitude $\underset{\sim}{E}_0$ to be vectors:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{ \underset{\sim}{\vec{E}}_0 \exp\left[i\left(\vec{k} \cdot \vec{r} - \omega t \right) \right] \right\}$$

The complex vector amplitude has six numbers that must be specified to completely determine it!

$$\underset{\sim}{\vec{E}}_0 = (\underbrace{\mathrm{Re}\{E_x\} + i\,\mathrm{Im}\{E_x\}}_{x\text{-component}},\ \underbrace{\mathrm{Re}\{E_y\} + i\,\mathrm{Im}\{E_y\}}_{y\text{-component}},\ \underbrace{\mathrm{Re}\{E_z\} + i\,\mathrm{Im}\{E_z\}}_{z\text{-component}})$$

Boundary Conditions

Often, a wave is constrained by external factors, which we call **Boundary Conditions**.

For example, a guitar string is attached at both ends.

In this case, only certain wavelengths/frequencies are possible.

Here the wavelengths can be:

λ_1, $\lambda_1/2$, $\lambda_1/3$, $\lambda_1/4$, etc.