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Fourier Series, Examples and the Fourier
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I. SYNOPSIS

The Fourier Integral is introduced by converting a Fourier series, in complex form, into the integral. Some examples are then given.

II. INTRODUCTION

We chose to introduce Fourier Series using the Particle in a Box solution from standard elementary quantum mechanics, but, of course, the Fourier Series antedates Quantum Mechanics by quite a few years (Joseph Fourier, 1768-1830, France).

Normal discussion of Fourier Series starts with a domain for the independent variable (here x) from $-\pi \leq x \leq \pi$ and considers replicating functions (such as sine and cosine) which map partly on this domain, and yet really extend over the domain $-\infty \leq x \leq +\infty$, replicating themselves every 2π .

So, assume we have a function $f(x)$ in the domain $-\pi \leq x \leq \pi$ which may be replicating itself as noted above. The Fourier Series for $f(x)$ is then given by

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{n=\infty} (A_n \sin nx + B_n \cos nx) \quad (2.1)$$

To repeat the derivation of the minimum error (above) here would require us to come to grips with the idea that $\sin x$ and $\cos x$ are orthogonal to each other. These integrals are trivial, over the domain in question, whether using 'x' or 'nx'. All one really needs is DeMoivre's Theorem and some familiarity in using it.

The coefficients are determinable (using this orthogonality) via

$$A_n = \frac{\int_{-\pi}^{\pi} f(x) \sin nxdx}{\int_{-\pi}^{\pi} \sin^2 nxdx} \quad (2.2)$$

$$B_n = \frac{\int_{-\pi}^{\pi} f(x) \cos nxdx}{\int_{-\pi}^{\pi} \cos^2 nxdx} \quad (2.3)$$

and

$$A_0 = \frac{\int_{-\pi}^{\pi} f(x)dx}{\int_{-\pi}^{\pi} dx} \quad (2.4)$$

(where the last term is often thought of as the "direct current" equivalent value, when translating the original $f(x)$ into a language of voltage (f) versus time (x)).

Anyway, assuming that we accept the 2π domain Fourier Series, can we go on to any "even" domain Fourier

Series? Yes. Consider the domain $-L/2 \leq \tau \leq +L/2$, and write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} \left\{ a_n \sin \left(\frac{2n\pi x}{L} \right) + b_n \cos \left(\frac{2n\pi x}{L} \right) \right\} \quad (2.5)$$

where we use lower case letters for the constants in this case. The same argument that got us the coefficients before, works here, and we find

$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin \left(\frac{2n\pi x}{L} \right) dx \quad (2.6)$$

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos \left(\frac{2n\pi x}{L} \right) dx \quad (2.7)$$

with

$$a_0 = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin \left(\frac{0\pi x}{L} \right) dx = \frac{1}{L} \int_{-L/2}^{L/2} f(x) dx \quad (2.8)$$

Converting to Complex Form

We now use DeMoivre's Theorem, to re-write these equations in a form more suited to the purposes at hand, i.e., passing to the Fourier Integral.

$$a_n = \frac{1}{2L} \int_{-L/2}^{L/2} f(x) \frac{e^{i2n\pi x/L} - e^{-i2n\pi x/L}}{i} dx \quad (2.9)$$

$$b_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \frac{e^{i2n\pi x/L} + e^{-i2n\pi x/L}}{2} dx \quad (2.10)$$

with

$$a_0 = \frac{1}{L} \int_{-L/2}^{L/2} f(x) dx \quad (2.11)$$

Adding, we have

$$c_n = b_n + ia_n = \frac{1}{2L} \int_{-L/2}^{L/2} f(x) e^{i2n\pi x/L} dx \quad (2.12)$$

and, subtracting, we have

$$c_n^* = b_n - ia_n = \frac{1}{2L} \int_{-L/2}^{L/2} f(x) e^{-i2n\pi x/L} dx \quad (2.13)$$

so that our Fourier Series can be rewritten as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \sin \frac{2n\pi x}{L} + b_n \cos \frac{2n\pi x}{L} \right) = \sum_{n=-\infty}^{\infty} \left(c_n e^{i2n\pi x/L} \right) \quad (2.14)$$

where we have changed the summation from running $-\infty \rightarrow -1$ and then $a_0/2$ and then $1 \rightarrow -\infty$ to $-\infty \rightarrow +\infty$. Remember,

$$c_n = \frac{1}{2L} \int_{-L/2}^{L/2} f(x) e^{-i2\pi n x/L} dx \quad (2.15)$$

but this is deceptive (for future work) because of the presence of 'x' in the integral. One needs to remember that 'x' in this integral has disappeared once the integration

is done! That means that 'x' is a "dummy" variable. Which in turn means that we can re-write this equation as

$$c_n = \frac{1}{2L} \int_{-L/2}^{L/2} f(\tau) e^{-i2\pi n \tau/L} d\tau$$

Combining the last two equations into a single one we have

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2L} \int_{-L/2}^{L/2} f(\tau) e^{i2\pi n \tau/L} d\tau \right) e^{-i2n\pi x/L} \right) \quad (2.16)$$

$$\begin{aligned} &= \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2L} \int_{-L/2}^{L/2} f(\tau) e^{i2\pi n \tau/L} d\tau \right) e^{-i2n\pi x/L} \right) (n+1-n) = \\ &\quad \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2L} \int_{-L/2}^{L/2} f(\tau) e^{i2\pi n \tau/L} d\tau \right) e^{-i2n\pi x/L} \right) \Delta n \end{aligned} \quad (2.17)$$

which is, passing through the standard calculus procedure

$$f(x) = \int_{n=-\infty}^{\infty} \left(\left(\frac{1}{2L} \int_{-L/2}^{L/2} f(\tau) e^{i2\pi n \tau/L} d\tau \right) e^{-i2n\pi x/L} \right) dn \quad (2.18)$$

$$f(x) = \frac{1}{2L} \int_{n=-\infty}^{\infty} \left(\int_{-L/2}^{L/2} f(\tau) e^{i2\pi n \tau/L - i2n\pi x/L} \right) d\tau dn \quad (2.19)$$

Defining $\omega = 2\pi n/2L$ we have

$$d\omega = \pi/L dn \quad (2.20)$$

so, solving, we have

$$dn = \frac{1}{\pi/L} d\omega \quad (2.21)$$

so

$$f(x) = \left(\frac{1}{2L} \right) \left(\frac{1}{\pi/L} \right) \int_{\omega=-\infty}^{\infty} \left(\int_{\tau=-L}^{\tau=L} f(\tau) e^{i\omega(\tau-x)} \right) d\tau d\omega \quad (2.22)$$

which, in the limit, $L \rightarrow \infty$ becomes

$$f(x) = \frac{1}{4\pi} \int_{\omega=-\infty}^{\omega=\infty} \left(\int_{\tau=-\infty}^{\tau=\infty} f(\tau) e^{i\omega(\tau-x)} \right) d\tau d\omega \quad (2.23)$$

We can split this into symmetric "pairs":

$$f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\omega=\infty} (g(\omega) e^{i\omega\tau}) d\omega \quad (2.24)$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\lambda=-\infty}^{\lambda=\infty} (f(\tau) e^{-i\omega\tau}) d\tau \quad (2.25)$$

III. AN EXAMPLE

Consider the Fourier Transform of the function $A \cos \omega_0 \tau$ in a range $-a \leq \tau \leq a$ and zero elsewhere. This is a truncated cosine! We have

$$g(\omega) = \int_{-a}^a \frac{1}{\sqrt{2\pi}} A \cos \omega_0 \tau e^{-i\omega\tau} d\tau$$

where the limits are recognizing that $f(\tau)$ (the cosine) is zero outside the domain $|\tau| \geq a$, which is easily evaluated using DeMoivre's Theorem (smile):

$$\cos \omega_0 \tau = \frac{e^{i\omega_0 \tau} + e^{-i\omega_0 \tau}}{2}$$

so we have

$$g(\omega) = \frac{1}{\sqrt{2\pi}} A \int_{-a}^a \left(\frac{e^{i\omega_0 \tau} + e^{-i\omega_0 \tau}}{2} \right) e^{-i\omega\tau} d\tau$$

$$g(\omega) = \frac{A}{2\sqrt{2\pi}} \left\{ \frac{e^{i(\omega_0 - \omega)a}}{i(\omega_0 - \omega)} - \frac{e^{-i(\omega_0 - \omega)a}}{i(\omega_0 - \omega)} + \frac{e^{-i(\omega_0 + \omega)a}}{-i(\omega_0 + \omega)} - \frac{e^{+i(\omega_0 + \omega)a}}{-i(\omega_0 + \omega)} \right\}$$

$$g(\omega) = \frac{A}{\sqrt{2\pi}} \left\{ \frac{\sin(\omega_0 - \omega)a}{(\omega_0 - \omega)} - \frac{\sin(\omega_0 + \omega)a}{(\omega_0 + \omega)} \right\}$$

IV. THE CLASSIC EXAMPLE

The classic Fourier Transform illustration is the Gaussian, since the transform of a Gaussian turns out to be a Gaussian itself. Then as one narrows one Gaussian, the other widens, illustrating the Heisenberg uncertainty principle relating 'x' to 'p' expectation values. We have

$$g(\omega) = A e^{-B\omega^2}$$

defining a function g which is Gaussian in its dependence. We then have

$$f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\omega=\infty} (g(\omega) e^{i\omega\tau}) d\omega$$

which would read, substituting $g(\omega)$,

$$f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\omega=\infty} \left(A e^{-B\omega^2} e^{i\omega\tau} \right) d\omega$$

How do we do this integral? Recalling from high school the "completing the square" idea, we write

$$B \left(\omega^2 - i \frac{\omega\tau}{B} \right) = B (\omega + \alpha)^2 = B (\omega^2 + 2\omega\alpha + \alpha^2)$$

so,

$$i \frac{\omega\tau}{B} = 2\omega\alpha$$

which becomes, upon integration

$$g(\omega) = \frac{A}{2\sqrt{2\pi}} \int_{-a}^a \left(\frac{e^{i(\omega_0 - \omega)\tau} + e^{-i(\omega_0 + \omega)\tau}}{2} \right) d\tau$$

which becomes, upon integration

$$g(\omega) = \frac{A}{2\sqrt{2\pi}} \left(\frac{e^{i(\omega_0 - \omega)\tau}}{i(\omega_0 - \omega)} \Big|_{-a}^a + \frac{e^{-i(\omega_0 + \omega)\tau}}{-i(\omega_0 + \omega)} \Big|_{-a}^a \right)$$

must be true, identifying α in terms of problem variables.

$$\alpha = \frac{i\tau}{2B}$$

and, of course

$$\alpha^2 = \left(\frac{i\tau}{2B} \right)^2$$

This means that the integrand in question can be written as

$$e^{-B \left((\omega + \frac{i\tau}{2B})^2 - \frac{\tau^2}{4B^2} \right)}$$

Substituting, we have

$$f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\omega=\infty} \left(A e^{-B \left((\omega + \frac{i\tau}{2B})^2 - \frac{\tau^2}{4B^2} \right)} \right) d\omega$$

which allows us to move the constant factor out of the integrand, obtaining

$$f(\tau) = e^{-B \frac{\tau^2}{4B^2}} \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\omega=\infty} \left(A e^{B \left((\omega + \frac{i\tau}{2B})^2 \right)} \right) d\omega$$

i.e.,

$$\gamma = (\omega + i\tau/(2B))$$

and

$$d\gamma = d\omega$$

So

$$f(\tau) = Ae^{-B\frac{\tau^2}{4B^2}} \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\omega=\infty} e^{-B\gamma^2} d\gamma$$

Now, it is known that the Gauss integral:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

where here $x=\gamma$ and $a=B$, so

$$f(\tau) = \frac{A}{i} e^{-B\frac{\tau^2}{4B^2}} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{B}}$$

or, cleaning up,

$$f(\tau) = \frac{A}{i\sqrt{2B}} e^{-\frac{\tau^2}{4B}}$$